Money, Capital and Unemployment

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Abstract

We study the role played by trading frictions in the long-run relationship between inflation, unemployment and capital accumulation. We model trading frictions via a positive endogenous probability of not finding a trading partner, and emphasize the implications this has for firms when making capital decisions prior to the opening of the two frictional markets: the labour market and the goods market. The economy is populated with firms and households. Firms borrow money from households to buy capital, then hire labour and finally pay wages, dividends and interests on loans. Households earn wages by working for firms, interests by lending to firms and dividends through firms ownership. Since firms make their capital choice prior to the opening of the frictional goods and labour markets, this creates an endogenous systemic risk of default and multiple equilibria. We characterize these equilibria, optimal monetary and fiscal policy and calibrate the model to the U.S. data.

Keywords: Money, Inflation, Unemployment, Capital.
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1 Introduction

This paper examines the role played by trading frictions in the long-run equilibrium relationship between inflation, unemployment and capital accumulation. By trading frictions we mean features of the economy that make it non-Walrasian, such as the difficulty to find a suitable trading partner, asymmetric information or coordination failures. In this paper we concentrate on the first one, and model it as an endogenous probability of not being able to trade on two frictional markets: the labour market and the goods market.

Trading frictions have attracted a lot of attention since the seminal work of Stigler (1961). The role of frictions in the formation and destruction of matches in the labour market is now well understood (Pissarides (2000); Mortensen (2003)). More frictions on the job market decreases labour market efficiency and increases unemployment. For the goods market, modelling trading frictions has not only made it possible to price money positively without the help of ad-hoc assumptions (Shi (1997); Lagos and Wright (2005)), but also clarified the effect of inflation on trade (Berentsen, Rocheteau and Shi (2007)). In all these models trading frictions reduce the volume of trade, either through the quantity traded (intensive margin) or though the number of trades (extensive margin).

Though important, we believe that this *lower-trade effect* of frictions is only part of the story. If firms have to borrow money to buy the capital they need, and if frictions make it uncertain whether they will find a worker or a customer, then capital decisions by firms become risky ones. What if a firm is unable to find a customer, and therefore unable to pay back the loan and the wages? What if a firm is not even able to find a worker? Incorporating a capital decision into a model with frictions (in both the labor market and the goods market) generates a risk of default for firms with three important effects for the economy: First, unmatched firms will disappear from the economy, yet they can be replaced via free entry; second, workers working for those unmatched firms will not be paid and will not get their financial investment back; finally, the total dividends paid to households at the end of the trading period will be reduced in accordance with the number of failed businesses. Because households cannot reduce this risk of default by diversification, we call it *systemic risk*.

There exists several models that combine frictions on more than one market, or try to link frictions on one market to the operating of other Walrasian markets. For instance, Pissarides (2000) studies an extension of his model of frictional unemployment in which firms have to decide for their level of capital. Neither trade on the goods market nor money are explicitly modelled however, and capital decisions are taken once the firm has hired a worker. The absence of these ingredients eliminate the risk of default (Pissarides (2000) p. 24). Shi (1999), Aruoba and Wright (2003), and Aruoba, Waller and Wright (2007) examine how frictions on the goods market impact on capital decisions. However, they do not distinguish between firms and households so that any non-used capital can be consumed in the end by the agent, also eliminating the risk of default. Finally, Berentsen, Menzio and Wright (2007) study the interaction between labour market frictions, goods market frictions, and monetary policy. Although there is no capital in their model, there is the risk a firm cannot pay wages if unmatched with a customer. They eliminate this risk by opening a second-chance Walrasian market on which firms sell leftover output. By contrast, in this paper, the consequences of default for households and the markets are explicitly modelled. To model this risk and study its implications, we need two ingredients in addition to simply adding capital to frictional labour and goods markets. First, we need to separate agents between households and firms so that failure to payback has repercussions beyond the unlucky agent. Second, firms have to make capital decisions (and then to borrow the corresponding amount of money) prior to knowing whether they will find a suitable worker and a customer.

The goal of this paper is to provide a simple set of relationship between unemployment, inflation and the accumulation of capital by firms, and hopefully be able to account for the observed long-run correlations between those three blocks of the macroeconomy. We also aim at coming back on some of the main issues in the macroeconomics of growth and money. For instance, since inflation constitutes a tax on money holdings, households are expected to move away from money towards capital at times of inflation, a mechanism known as the Tobin effect (Tobin (1965)). Since this effect has little empirical support (e.g., Walsh (2001), ch.2), it has been argued that whether a Tobin effect is at work, or it is dominated by other effects. In our model, more inflation means lower activity, higher unemployment and therefore less capital needed, offsetting the Tobin effect in a way consistent with empirical evidence. Also, we expect the introduction of an endogenous default probability to generate rich feed-back between the capital market, the labour market, and the goods market, and possibly multiple equilibria depending on the equilibrium value of the default probability. Once these long-run relationships are understood, we move to the study of optimal fiscal and monetary policy. Despite recent efforts in these direction (e.g., Aruoba and Chugh (2006)), Kocherlakota (2005) points out that the money search literature does not consider policy instruments beyond the inflation tax, and also ignores the existence of other assets and the impact of monetary policy on these other assets (see, however, Geromichalos, Licari and Suárez-Lledó (2007) and Lester, Postelwaite and Wright (2007)). The interaction between the frictions and the three markets may generate new recommendations as to which tax/monetary policy to implement to foster long-term growth and low unemployment.

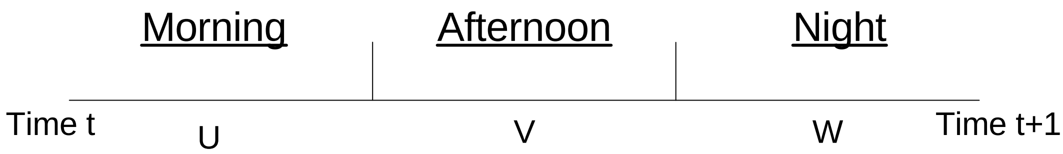
The paper is organized as follows. Section 2 describes the environment and the agents' decision problems. In Section 3 we characterize the equilibria. Numerical analysis is conducted in Section 4 to see how good the model is at replicating the U.S. business cycle key features. The last section concludes.

2 The Model

The economy is populated with two types of agents, *households* and *firms*. We index them by h and f, respectively. The measure of households is one, and the measure of firms is arbitrarily large, although not all firms will be active at any point in time. Households work for firms, lend money to firms, consume and save. Firms produce using labour and capital. Firms borrow money from households to build capital, hire workers to whom they pay wages. In addition, firms pay interests (on loans) and dividends to households.

Time is discrete and the horizon is infinite. Each period is divided into three subperiods, say morning, afternoon and night, during which the market structure and economic activity differ. Agents discount between periods at rate $\beta \in (0,1)$, but not between subperiods within a period. As shown in Figure 1, we introduce three value functions for the three markets, U^j, V^j and W^j, respectively, with $j = h,\ f$.

Figure 1 Timing



Money is intrinsically useless, perfectly divisible and storable. The gross growth rate of the money supply at date t is γ_t; that is, $M_{t+1} = \gamma_t M_t$, where M_t is the quantity of money per household at the beginning of period t. New money is injected by lump-sum transfers to households in the last subperiod.

The last subperiod: Several activities take place in the last subperiod. Households are endowed with a vector of endowments $\vec{x}$, which they can consume in quantity x. These endowments, however, can also be sold to firms as a necessary input (raw materials) to produce the capital firms need. Firms are the only agents with the ability to turn endowments into capital. Noting $y = \vec{x} - x$ the leftover endowments, capital is produced by firms according to the technology

$$k = j(y),$$

where the function j satisfies the usual properties: $j(0) = 0$, $j' > 0$ and $j'' < 0$. These leftover endowments are sold to firms on a Walrasian market to operate at the end of this last subperiod. In this paper, we take it seriously the idea that capital is transformed from raw materials, and capital is essentially a real good. We adopt this approach to guarantee that capital will not compete with fiat money as medium of exchange.

Prior to this market for endowments, there is a financial market on which firms (who have just paid the wages, interests and dividends to households), can borrow money from households, money they will need in the coming market to buy endowments from households. While firms decide how much to borrow (and hence how much endowment to purchase), households decide how much of their wealth to lend to firms and how much to keep in the form of money. This supply of loanable funds coming from households and the demand for loanable funds coming from firms equalize on the Walrasian financial market determining the real interest rate in the economy. Moreover, all incomes due by firms to households (wages, interests and dividends) are paid at the opening of this last subperiod.

A key feature of the model is that some firms will go bankrupt and hence will not be able to meet their payment commitments. Basically, a firm will be able to pay out wages and interests on loans if and only if the firm is matched on the job market as well as on the goods market, an event that happens with probability ψ to be defined below. In other words, any firm that is unmatched on either the job market or the goods market will have no revenues, and will have to go bankrupt. This generates a flow of firms out of the economy: a portion $1-\psi$ of firms exit the economy every period. We call ψ the *survival rate* for firms. This also generates a risk of default, which enters the budget constraint of the representative household. Apparently, bankruptcy of firms is welfare costly in that it destroys the resources (labor, capital) that firms borrowed from households.

The first subperiod: The labor market that opens in the first subperiod is a standard Mortensen-Pissarides labor market. In that market, existing jobs are destroyed at an exogenous rate δ. New firms enter the market at a cost and post vacancies, and are matched bilaterally with unemployed households at random. The probability for a household to meet a firm is $\lambda^h = L(u,v)/u$, where u is the measure of unemployed households and v is the measure of vacancies posted by firms. As is standard, the matching function L has constant return to scales, and thus $\lambda^h = L(1, v/u)$. Likewise, the probability for a firm to meet a household is $\lambda^f = L(u/v, 1)$.

If a firm and a worker meet, they bargain over the wage and sign a contract. This contract stipulates that (1) the firm will use the worker's labour force if the firm pairs with a customer in the goods market. The nominal wage w is determined via Nash bargaining between the firm and the household. Although wages are determined in the first subperiod, but they are not actually paid until the last subperiod; (2) if the firm is not matched with a customer, then no production will take place. As the firm will go bankrupt, no wage will be paid in this case.

We want to emphasize that unemployment exists in the economy because during the matching process some vacancies will not find any worker (an event that happens with endogenous probability $1 - \lambda^f$), and some of the existing jobs will be destroyed at the exogenous rate δ. Both events create unemployment.

The second subperiod: We model the goods market in the spirit of a Lagos-Wright framework: households and employed firms are matched pairwise, and trade is bilateral. We assume that in these bilateral situations, agents bargain according to the generalized Nash protocol over the quantity of goods (q) and money to be exchanged (d). The probability for a firm to meet a customer is $\alpha^f = M(B, S)/S$, where B and S are the measures of households and firms in the goods market. An important characteristic of the matching function M is constant returns to scales, which implies that $\alpha^f = M(B/S, 1)$. While all households participate in the decentralized goods market, only employed firms produce in the goods market. Thus, $B = 1$, $S = (1 - u)(1 - \delta) + v\lambda^f$, and $\alpha^f = M(1/((1 - u)(1 - \delta) + v\lambda^f), 1)$. Likewise, the probability for an household to find a firm is $\alpha^h = M(B, S)/B = M(1, (1 - u)(1 - \delta) + v\lambda^f))$.

We can now compute the *survival rate* for firms

$$\psi = \left[(1 - u)(1 - \delta) + v\lambda^f\right] \alpha^f.$$

2.1 Households

As in the standard Mortensen-Pissarides model, we let e denote *employment status*: $e = 1$ indicates that a household is matched with a firm in the labor market; $e = 0$ indicates otherwise.

In addition, we let s denote *matching status* of the firm that the household has signed a wage contract with: $s = 1$ indicates the firm has successfully found a consumer in the previous goods market; $s = 0$ indicates otherwise.

Let $W^h_{e,s}$ denote the household's expected payoff from entering the last subperiod with m units of money and a units of financial assets. A representative household chooses consumption of good x, lends $\hat{a}$ to the firms in the financial market, and brings money balances $\hat{m}$ into the next period, to solve

$$W^h_{e,s}(m, a) = \max_{x,\hat{m},\hat{a}} \left\{ x + \beta \hat{U}^h_s(\hat{m}, \hat{a}) \right\} \tag{1}$$

$$s.t.\ \hat{m} + \hat{a} + px = p\vec{x} + \Delta + \psi\,(1 + r)\,a + esw + m + \tau,$$

where $\vec{x}$ is the household's endowment, Δ is dividend income, p is the price of x, r is the real interest rate, and τ is the lump-sum transfer from the central bank. For money to grow at a constant rate γ, the lump-sum transfer must satisfy $\tau = (\gamma - 1)M$. For notational ease we use a hat over a variable to denote the value of the variable in the next period.

Whether it signed a contract or not in the previous job market, a household still has resources via his endowments, interests on loans to firms and dividends paid by firms. Whether it signed a contract or not, a household need to decide how much money to bring along for shopping in the decentralized goods market and how much to lend to firms. It is clear from the household's budget constraint that the default risk of firms affect households' wealth in two ways: first, the household will have wage income if and only if it is employed ($e = 1$) and the firm that the household has signed a contract with has a successful match in the previous goods market ($s = 1$). Second, due to the default risk, the rate of return to lend to firms is $\psi\,(1 + r)$.

As can be seen, we assume quasi-linearity in the utility of consuming these endowments, which implies the optimal choice of ($\hat{m}$, $\hat{a}$) is independent of (m, a). As a result, the distribution of ($\hat{m}$, $\hat{a}$) is degenerate at the beginning of the following period.[1]

Let $U^h_1(m, a)$ be the value function for an employed household entering the labour market,

[1] For details, refer to Lagos and Wright (2005).

and $U_0^h(m,a)$ be the value function of an unemployed household. Thus,

$$U_1^h(m,a) = \delta V_0^h(m,a) + (1-\delta)\, V_1^h(m,a), \text{ and}$$

$$U_0^h(m,a) = \lambda^h V_1^h(m,a) + \left(1-\lambda^h\right) V_0^h(m,a).$$

Note that by contrast to firms, separated or unmatched households do not exit the economy. They simply proceed to the goods market knowing that they will not received any salary in the last subperiod.

An unemployed household entering the goods market with money holding m and financial assets a has expected lifetime utility

$$V_0^h(m,a) = \alpha^h \left[u(q) + W_{0,0}^h(m-d,a)\right] + \left(1-\alpha^h\right) W_{0,0}^h(m,a),$$

where the utility function u is concave, with $u(0) = 0$, and $u'(0) = \infty$. With probability α^h, the household meets a firm and trade takes place. As a result, the firm produces, the household consumes q, and makes a nominal payment d to the firm. In contrast, for an employed household, with probability $1-\alpha^f$ the firm that the household has signed a contract with will go bankrupt, and so the household will not receive any salary in the last subperiod. Thus,

$$\begin{aligned} V_1^h(m,a) \quad &= \quad \alpha^h \left[u(q) + \alpha^f W_{1,1}^h(m-d,a) + \left(1-\alpha^f\right) W_{1,0}^h(m-d,a)\right] \\ &\quad + \left(1-\alpha^h\right) \left[\alpha^f W_{1,1}^h(m,a) + \left(1-\alpha^f\right) W_{1,0}^h(m,a)\right]. \end{aligned}$$

2.2 Firms

A firm that has been successful in both finding a worker and a customer enters the last subperiod. The firm enters the last subperiod with cash receipts m and capital stock k. In the last subperiod, existing firms adjusts their capital stocks (repays old loans, borrow new loans), pays wages, real interests on previous period capital, and dividends to households. Apparently, firms do not need money in either the first or the second subperiods. For simplicity, we assume capital completely depreciate between two periods. The existing firm's problem is

$$W_1^f(m,k) = \max_{\hat{k}} \; \frac{m}{p} - \frac{w}{p} - (1+r)\, j^{-1}(k) + \beta \hat{U}_1^f(\hat{k}),$$

where $\hat{k}$ is the firm's demand for capital for the next period.

We assume free entry of firms: firms considering participating to the economy (and then known as new firms) decides whether to pay a real cost of l to enter the labor market with a vacancy that might match with an household. New firms that decide to enter the labor market then choose next period capital stock to maximize their expected profits. Even though there is no reason for existing and new firms to choose the same level of capital, we show below that they do so and we simply note this level $\hat{k}$. Thus,

$$W_n^f = \max\left\{0, -l + \max_{\hat{k}} \beta \hat{U}_n^f(\hat{k})\right\}.$$

On the labour market existing firms are separated from their workers at rate δ and new firms are matched with workers at rate λ^f so that

$$U_1^f(k) = (1-\delta)\, V_1^f(k), \text{ and}$$

$$U_n^f(k) = \lambda^f V_1^f(k).$$

Finally, on the goods market a firm meets a customer with probability α^f and pays variable costs $c(q)$ so that

$$V_1^f(k) = \alpha^f \left[-c(q) + W_1^f(d,k)\right].$$

We assume the cost function c satisfies the usual assumptions, $c(0) = c'(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$. The cost function here captures the realistic feature that other than the cost of labor and capital, firms incur some variable costs in production, and these costs are proportional to the firm's output.

3 Equilibrium

Before proceeding to characterize an equilibrium, we introduce some convenient mathematical features of the model. First, due to the linearity of the household's value function $W_{e,s}^h(m,a)$, we could reduce the three value functions into one Bellman equation.

$$\begin{aligned} W^h_{1,1}(m,a) &= \beta\left\{\delta \hat{W}^h_{0,0}(0,0) + (1-\delta)\hat{\alpha}^f \hat{W}^h_{1,1}(0,0) + (1-\delta)(1-\hat{\alpha}^f)\hat{W}^h_{1,0}(0,0)\right\} + \frac{I_1}{p} \\ & \frac{m+\psi(1+r)a}{p} + \max_{\hat{m},\hat{a}}\left\{\beta\hat{\alpha}^h[u(\hat{q}) - \frac{\hat{d}}{\hat{p}}] - \hat{m}(\frac{1}{p} - \frac{\beta}{\hat{p}}) - \hat{a}[\frac{1}{p} - \frac{\beta\hat{\psi}(1+\hat{r})}{\hat{p}}]\right\}, \end{aligned}$$

$$\begin{aligned} W^h_{1,0}(m,a) &= \beta\left\{(1-\hat{\lambda}^h)\hat{W}^h_{0,0}(0,0) + \hat{\lambda}^h\hat{\alpha}^f \hat{W}^h_{1,1}(0,0) + \hat{\lambda}^h(1-\hat{\alpha}^f)\hat{W}^h_{1,0}(0,0)\right\} + \frac{I_0}{p} \\ & \frac{m+\psi(1+r)a}{p} + \max_{\hat{m},\hat{a}}\left\{\beta\hat{\alpha}^h[u(\hat{q}) - \frac{\hat{d}}{\hat{p}}] - \hat{m}(\frac{1}{p} - \frac{\beta}{\hat{p}}) - \hat{a}[\frac{1}{p} - \frac{\beta\hat{\psi}(1+\hat{r})}{\hat{p}}]\right\}, \text{ and} \end{aligned}$$

$$\begin{aligned} W^h_{0,0}(m,a) &= \beta\left\{(1-\hat{\lambda}^h)\hat{W}^h_{0,0}(0,0) + \hat{\lambda}^h\hat{\alpha}^f \hat{W}^h_{1,1}(0,0) + \hat{\lambda}^h(1-\hat{\alpha}^f)\hat{W}^h_{1,0}(0,0)\right\} + \frac{I_0}{p} \\ & \frac{m+(1+r)a}{p} + \max_{\hat{m},\hat{a}}\left\{\beta\hat{\alpha}^h[u(\hat{q}) - \frac{\hat{d}}{\hat{p}}] - \hat{m}(\frac{1}{p} - \frac{\beta}{\hat{p}}) - \hat{a}[\frac{1}{p} - \frac{\beta(1+\hat{r})}{\hat{p}}]\right\}, \end{aligned}$$

where $I_1 = p\vec{x} + \tau + \Delta + w$, and $I_0 = p\vec{x} + \tau + \Delta$.

It is straightforward to show that the demand for q are identical across households, regardless of their employment status. The first-order conditions are

$$\hat{m} : \beta\hat{\alpha}^h[u'(\hat{q})q'(\hat{m}) - \frac{\hat{d}}{\hat{p}}'(\hat{m})] = \frac{1}{p} - \frac{\beta}{\hat{p}}, \text{ and} \tag{2}$$

$$\hat{a} : \beta\hat{\alpha}^h[u'(\hat{q})q'(\hat{k})k'(\hat{a}) - \frac{\hat{d}}{\hat{p}}'(\hat{q})q'(\hat{k})k'(\hat{a})] = \frac{1}{p} - \frac{\beta\hat{\psi}(1+\hat{r})}{\hat{p}}. \tag{3}$$

In these expressions, $q'(\hat{m})$ reveals the relationship between households' money holding and the amount of goods traded, and $k'(\hat{a})$ reveals the marginal effect of households' supply of loanable funds on capital stock. Once we specify how q and d are determined in equilibrium, we can substitute for their derivatives in (2) and (3).

Second, with repeated substitution, the existing firm's maximization problem can be simplified into the following program:

$$V^f_1(k) = \alpha^f[\frac{d}{p} - \frac{w}{p} - c(q) - (1+r)j^{-1}(k)\frac{p_{-1}}{p}] + \beta\hat{\alpha}^f(1-\delta)\max_{\hat{k}} \hat{V}^f_1(\hat{k}).$$

The first-order condition with respect to $\hat{k}$ is

$$\left[\frac{\hat{d}'}{\hat{p}}(\hat{k}) - c'(\hat{q})\right] q'(\hat{k}) = (1+\hat{r}) j^{-1\prime}(\hat{k})\frac{p}{\hat{p}}, \tag{4}$$

Condition (4) equates the benefit to the cost of capital. As we mentioned earlier, a convenient feature of the model is that new firms' demand for capital are identical to existing firms', and, therefore, condition (4) also defines new firms' demand for capital.

We now characterize equilibrium. Here is an outline of what will follow. We begin by specifying how nominal wages (w) and terms of trade (q, d) are determined. We then use some properties of these solutions to simplify the households and firms' maximization problem. In particular, we derive the money demand function and conditions with respect to the demand and supply of capital. These three conditions together with the credit market clearing condition, the wage equation, the steady state condition for unemployment (the Beveridge curve), and the free-entry condition in the labor market define a steady-state monetary equilibrium.

3.1 The Generalized Nash Problem in the Labor Market

As we said, we assume in the labor market the nominal wage w solves the generalized Nash problem, with bargaining power for the households given by η and threat points given by continuation values. The household's payoff from being employed is $V_1^h(m,a)$ and threat points $V_0^h(m,a)$. Due to the linearity of $W^h(m,a)$, the surplus for household is $V_1^h(m,a) - V_0^h(m,a) = \alpha^f \frac{w}{p} + \alpha^f \beta(1-\delta-\hat{\lambda}^h)[\hat{\alpha}^f \hat{W}_{1,1}^h(0,0) + (1-\hat{\alpha}^f)\hat{W}_{1,0}^h(0,0) - \hat{W}_{0,0}^h(0,0)]$. Similarly, the firm's surplus is $V_1^f(k) = \alpha^f[\frac{d}{p} - \frac{w}{p} - c(q) - (1+r)j^{-1}(k)\frac{p_{-1}}{p}] + \alpha^f\beta(1-\delta)\max_{\hat{k}} \hat{V}_1^f(\hat{k})$. Notice that the current-period nominal wages w do not appear in either $\hat{\alpha}^f \hat{W}_{1,1}^h(0,0) + (1-\hat{\alpha}^f)\hat{W}_{1,0}^h(0,0) - \hat{W}_{0,0}^h(0,0)$ or $\hat{V}_1^f(\hat{k})$, so we take their values as given at this stage. Let A and B denote the equilibrium values of these two terms, respectively. The bargaining problem in the labor market is

$$\max_w \left[\frac{w}{p} + \beta(1-\delta-\hat{\lambda}^h)A\right]^\eta \left[\frac{d}{p} - \frac{w}{p} - c(q) - (1+r)j^{-1}(k)\frac{p_{-1}}{p} + \beta(1-\delta)B\right]^{1-\eta}.$$

The solution to this bargaining problem yields

$$\frac{w}{p} = \eta\left[\frac{d}{p} - \frac{w}{p} - c(q) - (1+r)j^{-1}(k)\frac{p_{-1}}{p} + \beta(1-\delta)B\right] - (1-\eta)\beta(1-\delta-\hat{\lambda}^h)A. \tag{5}$$

We will substitute the steady-state values of A and B into this equation after we derive the equilibrium solution.

3.2 The Generalized Nash Problem in the Goods Market

We now proceed to specify the bilateral bargaining problem in the DM. The household's surplus from trade is $u(q) + W^h_{1,s}(m-d,a) - W^h_{1,s}(m,a)$ for employed households, and $u(q) + W^h_{0,0}(m-d,a) - W^h_{0,0}(m,a)$ for unemployed households. The linearity of $W^h(m,a)$ implies that the household's surplus from trade, regardless of their employment status, is $u(q) - \frac{d}{p}$. The firm's surplus is $W^f_1(d,k) - c(q) = \frac{d}{p} - \frac{w}{p} - c(q) - (1+r)j^{-1}(k)\frac{p_{-1}}{p} + \beta\hat{U}^f_1(\hat{k})$. Hence, our bargaining solution is

$$\max_{(q,d)} [u(q) - \frac{d}{p}]^\theta [\frac{d}{p} - \frac{w}{p} - c(q) - (1+r)j^{-1}(k)\frac{p_{-1}}{p} + C]^{1-\theta} \tag{6}$$

$$\begin{aligned} s.t.\ (1)\ d &\leq m, \\ (2)\ q &\leq f(k), \end{aligned}$$

where θ is the household's bargaining weight, and C denotes the equilibrium value of $\beta\hat{U}^f_1(\hat{k})$. The production function f satisfies the usual properties: $f(0) = 0$, $f' > 0$, and $f'' < 0$. As in a standard Lagos-Wright model, constraint (1) is the household's liquidity constraint. It simply says the household can pay no more money than his money holdings. Here, introducing capital and neoclassical firms into the framework creates a new twist on the bilateral bargaining solution: Constraint (2) requires that the amount of goods traded should be constrained by the firm's production capacity.

To solve this problem, we first ignore both constraints. Then necessary and sufficient conditions for a solution are

$$q : \theta u'(q)[\frac{d}{p} - \frac{w}{p} - c(q) - (1+r)j^{-1}(k)\frac{p_{-1}}{p} + C] = (1-\theta)c'(q)[u(q) - \frac{d}{p}], \text{ and} \tag{7}$$

$$d : \theta[\frac{d}{p} - \frac{w}{p} - c(q) - (1+r)j^{-1}(k)\frac{p_{-1}}{p} + C] = (1-\theta)[u(q) - \frac{d}{p}]. \tag{8}$$

Thus $u'(q) = c'(q)$, or $q = q^*$. Let m^* denote the optimal money holding corresponding to q^*, and $m^* = p\left\{(1-\theta)u(q^*) + \theta\left[\frac{w}{p} + c(q^*) + (1+r)j^{-1}(k)\frac{p-1}{p} - C\right]\right\}$. Also, define k^* as the k such that $q^* = f(k^*)$. We claim the solution to (6) is

$$q = \begin{cases} q^* & if\ m \geq m^*, k \geq k^* \\ \tilde{q}(m) & if\ m < m^*, k \geq k^* \\ f(k) & if\ m \geq m^*, k < k^* \\ f(k) & if\ m < m^*, k < k^*, \end{cases} \qquad d = \begin{cases} m^* & if\ m \geq m^*, k \geq k^* \\ m & if\ m < m^*, k \geq k^* \\ \tilde{d} & if\ m \geq m^*, k < k^* \\ m & if\ m < m^*, k < k^*, \end{cases}$$

where $\tilde{q}(m)$ is the q that solves the first-order condition (7), and $\tilde{d}$ is the d that solves condition (8).

Proposition 1: *In any equilibrium with $\theta \in (0,1)$ and $\gamma \geq \max\{\beta, \beta\psi(1+r)\}$, it must be true that $q < q^*$, $m < m^*$, and $k \geq k^*$.*

This result is quite intuitive: The assumption that the bargaining weight θ is strictly positive but less than 1 plays a key role here. On one hand, as explained in Lagos and Wright (2005), this creates a holdup problem for the household. The fact that the firm will bargain away part of the trading surplus reduces the household's demand for money balances and hence q. As a result, households do not carry "enough" money balances into the goods market, but save "too much"; On the other hand, this certainly increases the firm's incentive to borrow (so that they could extract maximum benefits from the trade). Therefore, in equilibrium the output traded is inefficiently low, and firms hold extra capital stock.

It is a simple matter to check that in the extreme case of $\theta = 0$ or $\theta = 1$, either no households or no firms will participate in the goods market, and the equilibrium breaks down. Moreover, the assumption $\gamma \geq \max\{\beta, \beta\psi(1+r)\}$ is made to guarantee the existence of a monetary equilibrium. In a standard model without capital, we simply assume $\gamma \geq \beta$ to ensure there will be positive demand for money. Here, the fact that capital earns a positive expected rate of return $\psi(1+r)$ requires the return on money γ should be higher than the expected return of capital as well; otherwise, households will choose to invest all their wealth in the financial market, and no one is willing to hold money.

3.3 Equilibrium

We can now use the properties of the bargaining solution to simplify the households' and firms' maximization problem. Substituting $d = m$ into (7), the Nash bargaining solution becomes:

$$\theta u'(q)[\frac{m}{p} - \frac{w}{p} - c(q) - (1+r)j^{-1}(k)\frac{p_{-1}}{p} + C] = (1-\theta)c'(q)[u(q) - \frac{m}{p}]. \tag{9}$$

It is worthwhile to emphasize here that according to Proposition 1, it must be true that $q < f(k)$ in equilibrium, and, therefore, condition (9) always holds as an equilibrium condition. Condition (9) defines a unique relationship between q and the other two variables that we are interested in: m/p and k, and we can summarize the relationship as follows:

$$q = g(\frac{m}{p}, k). \tag{10}$$

The importance of condition (10) is that it enables us to express q in terms of m/p and k, and hence enables us to derive the derivatives of q with respect to m/p and k. Inserting $q'(m) = g_{\frac{m}{p}}(\frac{m}{p}, k)/p$, $\frac{m}{p}'(q) = 1/g_{\frac{m}{p}}(\frac{m}{p}, k)$, and $q'(k) = g_k(\frac{m}{p}, k)$ into first-order condition (2), (3) and (4), dropping all time indexes in what follows (as we focus on stationary equilibria in which real allocations are constant), we arrive at

$$u'(q)g_{\frac{m}{p}} = 1 + \frac{\gamma - \beta}{\beta\alpha^h}, \tag{11}$$

$$[u'(q) - \frac{1}{g_{\frac{m}{p}}}]g_k j'(\frac{a}{p_{-1}}) = \frac{\gamma - \beta\psi(1+r)}{\gamma\beta\alpha^h}, \text{ and} \tag{12}$$

$$\left[\frac{1}{g_{\frac{m}{p}}} - c'(q)\right] g_k = \frac{1+r}{\gamma} j^{-1}(k), \tag{13}$$

where γ is the gross growth rate of money supply.

Condition (11) is a standard money demand function, as in Lagos and Wright (2005). It equates the marginal benefit and the marginal cost of acquiring money. Condition (12) and (13) defines the supply (a/p_{-1}) and demand (k) for capital, respectively. Notice this is a system with eight unknowns ($q, r, k, u, v, m/p, w/p, a/p_{-1}$) but only five equations so far (the wage equation (5), and equations (10) to (13)), therefore, we need three more equilibrium conditions to close the model.

First, recall that the free-entry condition in the labor market requires that $\max_{\hat{k}} \beta \hat{U}_n^f(\hat{k}) = l$, where l is the real cost that a new firm must incur to enter the labor market. Inserting the steady-state value of $\hat{U}_n^f(\hat{k})$, we arrive at

$$l = \frac{\beta\alpha^f \lambda^f}{1-\beta\alpha^f(1-\delta)} \left[\frac{m}{p} - \frac{w}{p} - c(q) - \frac{1+r}{\gamma} j^{-1}(k) \right]. \tag{14}$$

Second, the credit market-clearing condition equates the supply of capital (a/p_{-1}) to the demand for capital (k). Notice that the demand for capital comes from two types of firms: existing and new firms, and the measure of each type of firms is ψ and v, respectively. Hence, the market-clearing condition becomes

$$j(\frac{a}{p_{-1}}) = (\psi + v)k, \tag{15}$$

where function j represents the technology transforming raw materials into capital goods.

Finally, note that the so-called Beveridge curve (i.e., the steady-state condition for unemployment) will allow us to express v in terms of the unemployment rate u. Thus,

$$N(u, v) = (1-u)\delta. \tag{16}$$

To close this section, recall that so far we have taken the values of $\hat{\alpha}^f \hat{W}_{1,1}^h(0,0) + (1 - \hat{\alpha}^f)\hat{W}_{1,0}^h(0,0) - \hat{W}_{0,0}^h(0,0)$, $\hat{V}_1^f(\hat{k})$, and $\beta\hat{U}_1^f(\hat{k})$ as given. We now solve for the steady-state values of these terms as follows:

$$A \equiv \hat{\alpha}^f \hat{W}_{1,1}^h(0,0) + (1-\hat{\alpha}^f)\hat{W}_{1,0}^h(0,0) - \hat{W}_{0,0}^h(0,0) = \frac{\alpha^f}{1-\beta\alpha^f(1-\delta)} \frac{w}{p}, \tag{17}$$

$$B \equiv \hat{V}_1^f(\hat{k}) = \frac{\alpha^f}{1-\beta\alpha^f(1-\delta)} \left[\frac{m}{p} - \frac{w}{p} - c(q) - \frac{1+r}{\gamma} j^{-1}(k) \right], \text{ and} \tag{18}$$

$$C \equiv \beta\hat{U}_1^f(\hat{k}) = \frac{\beta\alpha^f(1-\delta)}{1-\beta\alpha^f(1-\delta)} \left[\frac{m}{p} - \frac{w}{p} - c(q) - \frac{1+r}{\gamma} j^{-1}(k) \right]. \tag{19}$$

Conditions (10) to (16) together with the wage condition (5) fully characterize a stationary monetary equilibrium, where the steady-state values of A, B and C are given by equations (17) to (19).

Definition: *A stationary monetary equilibrium consists of,*

(a) a set of prices $\{p, r, w\}$

(b) the household's decision $\{x, \hat{m}, \hat{a}\}$

(c) the firm's decisions $\{\hat{k}\}$

(d) the firm's dividend Δ

such that,

(1) given $\{p, r, w\}$*, the household's optimal plan solves the maximization problem (1),*

(2) given r*, the firm's capital demand solves the firm's profit maximization problem,*

(3) all markets clear,

(4) Δ *equals to the firm's net profit,*

(5) real allocations $\{q, x\}$ *are constant over time.*

4 Quantitative Analysis

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