LOCAL AND GLOBAL IDENTIFICATION OF DSGE MODELS: A SIMULTANEOUS-EQUATION APPROACH

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ABSTRACT. We address some issues about local and global identification of DSGE models and link these issues to identification in the simultaneous-equation VAR framework.

I. INTRODUCTION

Recently there has been an active line of research on identification of DSGE models (Canova and Sala, 2006; Fukac and Pagan, 2006; Cochrane, 2006; Nason and Smith, 2007; Beyer and Farmer, forthcoming; Iskrev, 2007). Most of the works deal with local identification only, partly because coefficients of the variables in DSGE models are nonlinear functions of model parameters. In this paper, we focus on the issues related to both local and global identifications. If a DSGE model can be determined to be identifiable a priori without looking at the data, there will be no need to go through an expensive likelihood search to discover this fact.

If a model is identified, there is an issue related to weak identification. Weak identification occurs when the likelihood at its peak is nearly flat in certain dimensions. One example is VAR models. As shown by (Waggoner and Zha, 1999), weak identification can be solved by using an informative prior such as the prior of Sims and Zha (1998). For the same reasons, one can effectively deal with weak identification in DSGE models with an informative prior as done in the recent DSGE literature. In this paper, therefore, we do not deal with weak identification, which is largely an empirical issue.

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Very preliminary The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Atlanta, the Federal Reserve System, or the Reserve Bank of New Zealand.

II. General Framework

If a DSGE model is invertible (Sims and Zha, 2006; Fernandez-Villaverde, Rubio-Ramirez, Sargent, and Watson, 2007), the solution to the model has the VAR form:

$$y_t' A_0 = \sum_{\ell=1}^{p} y_{t-\ell}' A_\ell + z_t' C + \varepsilon_t' \text{ for } 1 \leq t \leq T, \tag{1}$$

where the lag length p may be ∞ but in practice can be approximated by a large finite number. While the DSGE model implies cross-equation restrictions on the matrices A_ℓ's, most restrictions are exclusion restrictions. Let a be a vector of all free parameters in A_ℓ that satisfying those restrictions; and let θ be a vector of free parameters for the corresponding DSGE model that satisfy a priori restrictions. Then θ maps into a through a nonlinear function such that $a = f_\theta(\theta)$. A necessary condition for a DSGE model to be identified is that for any two parameter points $\theta^{(1)} \neq \theta^{(2)}$, $a^{(1)} \equiv f_\theta(\theta^{(1)} \neq a^{(2)} \equiv f_\theta(\theta^{(2)}$. This condition can be easily checked (Iskrev, 2007). For the rest of the paper, we assume that this necessary condition is always satisfied.

III. OLS, IV, and Simultaneous-Equation Approaches

Cochrane (2006) studies the following simple New-Keynesian model:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_{st}, \tag{2}$$

$$x_t = E_t x_{t+1} - \tau(R_t - E_t \pi_{t+1}) + u_{dt}, \tag{3}$$

$$R_t = \phi_\pi \pi_t + u_{Rt}, \tag{4}$$

where π_t, x_t, and R_t stand for inflation, output gap, and the nominal interest rate. For analytical tractability and clarity, we further assume that u_{st}, u_{dt}, and u_{Rt} are uncorrelated and i.i.d. that their distribution is Gaussian with mean zero and the variances σ_s^2, σ_d^2, and σ_R^2. For this model, Cochrane (2006) shows that when the unique equilibrium exists, if one regresses R_t on π_t, the estimate of ϕ_π will not depend on the true value of ϕ_π, which generates the data. In other words, ϕ_π is not identified by the OLS method.

In this model, since all three variables are simultaneously determined, there is no "policy instrument" for an instrument-variable (IV) estimation of the policy reaction equation (4), just as there is no "left-hand-side variable." As in the simultaneous-equation VAR literature, however, the policy parameter can still be locally identified using identifiable orthogonal shocks as predetermined variables (Hausman and Taylor, 1983). If so, the estimate of such a policy parameter can be carried out by the full-information maximum likelihood method.

For a unique equilibrium or an MSV equilibrium, we shall now show that the parameter β is not even locally identified but all the other parameters in the model (5) (including ϕ_π) are locally identified.

Denoting

$$y_t = \begin{bmatrix} \pi_t & x_t & R_t \end{bmatrix}',$$

we can write the equations (2)-(4) in the matrix form

$$y_t' \begin{bmatrix} \sigma_s^{-1} & 0 & -\phi_\pi \sigma_R^{-1} \\ -\kappa\sigma_s^{-1} & \sigma_d^{-1} & 0 \\ 0 & \tau\sigma_d^{-1} & \sigma_R^{-1} \end{bmatrix} = E_t y_{t+1}' \begin{bmatrix} \beta\sigma_s^{-1} & \tau\sigma_d^{-1} & 0 \\ 0 & \sigma_d^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \epsilon_{st} \\ \epsilon_{dt} \\ \epsilon_{Rt} \end{bmatrix}, \tag{5}$$

where ϵ_{st}, ϵ_{dt}, and ϵ_{Rt} has a standard normal distribution. It follows from the system (5) that the solution to this forward-looking model is

$$y_t' = \frac{1}{1+\kappa\tau\phi_\pi} \begin{bmatrix} \sigma_s & -\tau\phi_\pi\sigma_d & \phi_\pi\sigma_R \\ \kappa\sigma_s & \sigma_d & \kappa\phi_\pi\sigma_R \\ -\kappa\tau\sigma_s & -\tau\sigma_d & \sigma_R \end{bmatrix} \epsilon_t, \tag{6}$$

where

$$\epsilon_t = \begin{bmatrix} \epsilon_{st} \\ \epsilon_{dt} \\ \epsilon_{Rt} \end{bmatrix}.$$

To prove this result, one can simply substitute (6) into both sides of (5) and then verify that (5) holds. Since the discount factor β does not appear in the data-generating-process (6) implied by this NK model, this parameter is not identified at all.

To see how the rest of the parameters are locally identified, we rewrite (6) in the form of (1), which leads to the following simultaneous-equation SVAR form:

$$y_t' A_0 = \epsilon_t,$$

where

$$A_0 = \begin{bmatrix} a_{0,11} & 0 & a_{0,13} \\ a_{0,21} & a_{0,22} & 0 \\ 0 & a_{0,32} & a_{0,33} \end{bmatrix}, \tag{7}$$

with

$$a_{0,11} = \sigma_s^{-1},$$
$$a_{0,13} = -\phi_\pi \sigma_R^{-1},$$
$$a_{0,21} = -\kappa \sigma_s^{-1},$$
$$a_{0,22} = \sigma_d^{-1},$$
$$a_{0,32} = \tau \sigma_d^{-1},$$
$$a_{0,33} = \sigma_R^{-1}.$$

Clearly, any two different points of the vector of DSGE parameters (σ_s, σ_d, σ_R, ϕ_π, κ, and τ) imply two different points of the vector of SVAR parameters ($a_{0,11}$, $a_{0,13}$, $a_{0,21}$, $a_{0,22}$, $a_{0,32}$, and $a_{0,33}$). Thus, our necessary condition for identification is satisfied. As argued by Hausman and Taylor (1983) and Sims and Zha (1999) and formally proven by Rubio-Ramírez, Waggoner, and Zha (2007), every point of $a_{0,11}$, $a_{0,13}$, $a_{0,21}$, $a_{0,22}$, $a_{0,32}$, and $a_{0,33}$ is locally identified. Therefore, σ_s, σ_d, σ_R, ϕ_π, κ, and τ are locally identified. In this NK model, $\kappa \equiv \frac{(1-\beta\xi)(1-\xi_p)}{\xi}$ where ξ is the Calvo stickiness parameter. If the value of β is fixed a priori, ξ is also locally identified.

IV. Global Identification

In the DSGE literature, the existing results concern only local identification. When a DSGE model is locally identified, there does not exist any other *sufficiently close* point in the parameter space that gives the same likelihood. It is possible, however, that there exists another point (distinctly far from the original point) that gives the same likelihood. When this situation occurs, it is said that the model is not *globally* identified at the original point. This result is summarized by the following proposition.

Proposition 1. For a unique equilibrium or an MSV equilibrium, the NK model (5) is not globally identified for some parameters that have a positive measure.

To give a concrete example for Proposition 1, consider the parameter point with the following parameterization consistent with Lubik and Schorfheide (2004): $\phi_\pi = 2.0$, $\kappa = 0.58$, $\tau = 0.54$, $\sigma_d = 1.0$, $\sigma_s = 2.0$, and $\sigma_R = 0.2$. Using the algorithm described in Rubio-Ramírez, Waggoner, and Zha (2007), we obtain another distinct point in the parameter space that generates the same data dynamics. The parameter values at this point are: $\phi_\pi = 2.5014$, $\kappa = 0.9014$, $\tau = 0.5664$, $\sigma_d = 1.0242$, $\sigma_s = 2.4933$, and $\sigma_R = 0.2001$.

What we do not know at this point is whether the NK model is globally identified for some parameters that have a positive measure.

V. Other Models

V.1. **Policy responding to expected inflation.**

If we let policy respond to expected inflation instead of current inflation:

$$R_t = \phi_\pi E_t \pi_{t+1} + u_{Rt},$$

the response parameter ϕ_π and the discount factor β cannot be identified either locally or globally but all the other parameters are globally identifiable. To see why this is a case, one can show that the solution to this alternative NK model is

$$y_t' A_0 = \epsilon_t,$$

where

$$A_0 = \begin{bmatrix} \sigma_s^{-1} & 0 & 0 \\ -\kappa\sigma_s^{-1} & \sigma_d^{-1} & 0 \\ 0 & \tau\sigma_d^{-1} & \sigma_R^{-1} \end{bmatrix}.$$

Clearly, the parameters ϕ_π and the discount factor β play no role in generating the data. By Theorem 4 in Fernandez-Villaverde, Rubio-Ramirez, Sargent, and Watson (2007), A_0 implies that all the other parameters are identifiable.

V.2. **Identification through lags.**

If policy also responds to output:

$$R_t = \phi_\pi \pi_t + \phi_x x_t + u_{Rt},$$

then the unique solution to this NK model has the SVAR form

$$y_t' A_0 = \epsilon_t,$$

where

$$A_0 = \begin{bmatrix} \sigma_s^{-1} & 0 & -\phi_\pi\sigma_R^{-1} \\ -\kappa\sigma_s^{-1} & \sigma_d^{-1} & -\phi_x\sigma_R^{-1} \\ 0 & \tau\sigma_d^{-1} & \sigma_R^{-1} \end{bmatrix}.$$

All the parameters are not identified almost everywhere.

But identification can be restored by introducing lags. Consider the NK model studied by Lubik and Schorfheide (2004):

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_{st}, \tag{8}$$

$$x_t = E_t x_{t+1} - \tau(R_t - E_t \pi_{t+1}) + u_{dt}, \tag{9}$$

$$R_t = \rho_R R_{t-1} + (1-\rho_R)(\phi_\pi \pi_t + \phi_x x_t) + u_{Rt}. \tag{10}$$

For analytical tractability, we continue to assume that u_t are i.i.d. and Gaussian. In this case, there are 9 structural parameters and 9 moment conditions. By counting, this model may be locally identified. The proposition gives a very strong result.

Proposition 2. In the unique equilibrium, the canonical NK model (8)-(10) is globally identified.

Proof. See Appendix A □

V.3. **The choice of structural shocks matters.** Let us assume that we do not have observations for the output gap in the model (8) - (10). We can only observe the inflation rate, π_t, and the interest rate, R_t. Consequently, we wish to reduce the number of structural shocks to the number of observed variables. What shock shall we exclude? The following two propositions state that a proper choice may affect model identification.

Proposition 3. If the output gap x_t is unobservable and there is no demand shock, $\sigma^2_{u_d} = 0$, in the unique equilibrium, the canonical NK model (8)-(10) is locally identified.

Proof. See Appendix B □

Proposition 4. If the output gap x_t is unobservable and there is no monetary policy shock, $\sigma^2_{u_R} = 0$, in the unique equilibrium, the canonical NK model (8)-(10) is not identified.

Proof. See Appendix C □

VI. Economic identification

So far we have been concerned by identification issues in the econometric sense. But the SVAR identification methodology can also be used for economic identification (model identification, as in Preston, 1978). The economic theory does not prescribe a unique location of structural shocks. A time preference shock or habit shock in the Euler equation has effectively the same effect on household's consumption. Similarly, a cost-push shock or technology shock have the same impact on the inflation rate in the Phillips curve. The choice between these shocks is completely arbitrary, and it depends on individual preferences. In this section, we show that model identifiablity may help to guide such preferences.

[MF:] I think of using the model in Ireland (2004,RES) to make a point. He has four shocks: const-push, technology, preference and policy shocks; and three observable variables. He sudies which ones are the main drivers of a business cycle. I want to take the model and use the SVAR ident methodology for that.

APPENDIX A. PROOF OF PROPOSITION 2

The script of the proof is as follows.

The model (8)-(10) in the matrix form is

$$y_t' B_0 = y_{t-1}' C + E_t y_{t+1}' D + \epsilon_t, \tag{A1}$$

where

$$B_0 = \begin{bmatrix} \sigma_s^{-1} & 0 & -\phi_\pi \sigma_R^{-1} \\ -\kappa \sigma_s^{-1} & \sigma_d^{-1} & -\phi_x \sigma_r^{-1} \\ 0 & \tau \sigma_d^{-1} & \sigma_R^{-1} \end{bmatrix}, \quad D = \begin{bmatrix} \beta \sigma_s^{-1} & \tau \sigma_d^{-1} & 0 \\ 0 & \sigma_d^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho_R \sigma_R^{-1} \end{bmatrix}, \tag{A2}$$

and $\epsilon_t = \begin{bmatrix} \epsilon_{st} & \epsilon_{dt} & \epsilon_{Rt} \end{bmatrix}'$. It has the MSV representation

$$y_t' A_0 = y_{t-1}' A_1 + \epsilon_t, \tag{A3}$$

with A_0 and A_1 satisfying $A_0 = B_0 - A_1 A_0^{-1} D$, and $A_1 = C$. The matrix

$$A_0 = \begin{bmatrix} \sigma_s^{-1} & 0 & -\phi_\pi \sigma_R^{-1} \\ -\kappa \sigma_s^{-1} & \sigma_d^{-1} & -\phi_x \sigma_R^{-1} \\ a_{31}^0 & \tau \sigma_d^{-1} & \sigma_R^{-1} \end{bmatrix},$$

and

$$\beta = \frac{\sigma_s \left(1 + \phi_x \tau + \kappa \phi_\pi \tau + a_{31}^0 \phi_\pi \sigma_s\right) a_{31}^0}{\rho_R \left(a_{31}^0 \sigma_s + \kappa \tau\right)},$$

yields a unique and stable solution to the model. It follows that the necessary condition for model identification holds. All structural parameters appear in the data generating process, and any positive value of $\{\beta, \kappa, \tau, \rho_R, \phi_\pi, \phi_x, \sigma_s, \sigma_d, \sigma_R\}$ gives a unique value of A_0 and A_1.

To finish the proof, we use results from Rubio-Ramírez, Waggoner, and Zha (2007). We construct matrices $M_j(f(A_0, A_1))$ for $j = 1, 2, 3$, so that

$$M_1 = \left[\begin{array}{ccc} 0 & \sigma_s^{-1} & -\phi_\pi \sigma_R^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_R^{-1} \\ \hline 1 & 0 & 0 \end{array}\right], M_2 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_R^{-1} \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right], M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Because $rank(M_j) = 3$ for all $j = 1, 2, 3$, it follows from Theorem 2 in Rubio-Ramírez, Waggoner, and Zha (2007) that the model is globally identified.

APPENDIX B. PROOF OF PROPOSITION 3

The MSV representation to the model (8) - (10) is

$$A_0 y_t = A_1 y_{t-1} + F\varepsilon_t, \tag{A4}$$

where $y_t = [\pi_t, x_t, R_t]'$, $\varepsilon_t = [\varepsilon_{st}\varepsilon_{Rt}]$, and

$$A_0 = \begin{bmatrix} 1 & 0 & -\phi_\pi \\ -\kappa & 1 & -\phi_x \\ a^0_{31} & a^0_{32} & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho_R \end{bmatrix}, F = \begin{bmatrix} \sigma_s & 0 \\ 0 & 0 \\ 0 & \sigma_R \end{bmatrix}.$$

The measurement equation for this system of transition equations is

$$\xi_t = G y_t, \tag{A5}$$

where $\xi_t = [\pi^*_t, R^*_t]$ is the vector of observations on inflation, π_t, and the interest rate, R_t;

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Equations (A4) and (A5) constitute a state-space system. The next step is to find a VAR representation to the system – a transfer function. Substituting (A4) into (A5) and solving for ε_t, we obtain

$$\varepsilon_t = (GA_0^{-1}F)^{-1}\xi_t - (GA_0^{-1}F)^{-1}GA_0^{-1}A_1 y_{t-1}.$$

Substituting this expression to (A4) yields

$$y_t = A_0^{-1}F(GA_0^{-1}F)^{-1}\xi_t,$$

and substituting this further into the measurement equation (A5), we finally obtain a VAR representation to the original DSGE model

$$\bar{A}_0\xi_t = \bar{A}_1\xi_{t-1} + \varepsilon_t. \tag{A6}$$

$$\bar{A}_0 = (GA_0^{-1}F)^{-1} = \begin{bmatrix} \sigma_s^{-1} & -\phi_\pi\sigma_s^{-1} \\ (\kappa a^0_{32} + a^0_{31})\sigma_R^{-1} & (1+\phi_x a^0_{32})\sigma_R^{-1} \end{bmatrix}, \bar{A}_1 = GA_1F = \begin{bmatrix} 0 & 0 \\ 0 & \rho_R\sigma_R^{-1} \end{bmatrix}.$$

All structural parameters affect the data generating process. But we can see that only $(\sigma_x, \phi_\pi, \tau)$ can be identified. The NK model is not globally identified. However, if $(\beta(a^0_{31}), \rho_R, \phi_x)$ are predetermined, the rest of the parameters can be locally identified. To show that, we employ Theorem 2 in Rubio-Ramírez, Waggoner, and Zha (2007). Since

$$M_1 = \left[\begin{array}{cc} 0 & \rho_R \sigma_R^{-1} \\ \hline 1 & 0 \end{array} \right], M_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right].$$

are matrices of a full rank, the VAR is identified. Thus if $(\beta(a_{31}^0), \rho_R, \phi_x)$ are predetermined, we can backup unique values for $(\sigma_s, \phi_\pi, \sigma_R, \kappa, \tau(a_{32}^0))$, and the model is locally identified.

Appendix C. Proof of proposition 4

The proof follows the same logic as in Appendix B. Now the model contains a demand shock, but does not contain a monetary policy shock. Matrix F is

$$F = \left[\begin{array}{cc} \sigma_s & 0 \\ 0 & \sigma_d \\ 0 & 0 \end{array} \right].$$

Consequently, the SVAR matrices are

$$\bar{A}_0 = \left[\begin{array}{cc} \sigma_s^{-1} & -\phi_\pi \sigma_s^{-1} \\ -\frac{(\kappa a_{32}^0 + a_{31}^0)}{a_{32}^0} \sigma_R^{-1} & -\frac{(1+\phi_x a_{32}^0)}{a_{32}^0} \sigma_R^{-1} \end{array} \right], \bar{A}_1 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right].$$

Applying Theorem 2 in Rubio-Ramírez, Waggoner, and Zha (2007), we find that this system is not identified.

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